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On the analyticity properties of scaling functions in models of polymer collapse

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Abstract. We consider the mathematical properties of the generating and partition functions in the two-variable scaling region about the tricritical point in some models of polymer collapse. We concentrate on models that have a similar behaviour to that of interacting partially-directed self-avoiding walks (IPDSAW) in two dimensions. However, we do not restrict the discussion to that model. After describing the properties for a general class of models, and stating exactly what we mean by scaling, we prove the following theorem: If the generating function of finite-size partition functions has a tricritical cross-over scaling form around the θ-point, and the associated tricritical scaling function, $\hat{g}$, has a finite radius of convergence, then the partition function has a finite-size scaling form and importantly the finite-size scaling function, $\hat{f}$, is an *entire function*. In the IPDSAW case we have an explicit representation of the finite-size scaling function. We point out that given our description of tricritical scaling this theorem should apply *mutatis mutandis* to a wider class of θ-point models. We discuss the result in relation to the Edwards model of polymer collapse for which it has recently been argued that the finite-size scaling functions are *not* entire.

1. Introduction

There exist many exactly solvable examples of two-dimensional lattice models in statistical mechanics [1] where an expression for the thermodynamic-limit free energy and values for critical exponents at phase transition points can be calculated. However, only a few representations of either the thermodynamic (temperature–magnetic field), correlation function (temperature–distance), or finite-size (temperature–system size) scaling functions are known. The scaling functions for the correlation function [2], and the finite-size partition function and specific heat [3], of the two-dimensional Ising model are some examples. Recently, the scaling function of the generating function (grand partition function) around the tricritical point of the interacting partially-directed self-avoiding walk (IPDSAW) model [4] of polymer collapse was calculated‡. The generating function is written in terms of the variables temperature and monomer (step) fugacity and is the weighted sum of the finite-length partition functions. Therefore, it is an obvious task to transform this scaling function into one for the finite-size partition functions. Another reason to consider this scaling function is that some of the finite-size scaling functions of the standard continuum model of polymer collapse have recently [6] been argued to be non-analytic at zero argument. While

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‡ In this paper the scaling function for the case of the semicontinuous version of the model is found. It can now be demonstrated explicitly, with a straightforward modification of the work in [5] that the fully discrete model has the same scaling function.

our analysis will be concentrated on the IPDSAW case we shall state our central theorem in a general fashion. This is partly because we shall argue later that its conditions should apply to the more canonical interacting self-avoiding walk (ISAW) model (at least below the upper critical dimension).

We also write down in this paper (section 4) the set of standard tricritical scaling assumptions. These are translations of each of the physical ideas that characterizes a tricritical point (in the symmetry plane [7]). This allows us to set up the theorem concerning the finite-size scaling function that is proven later. It is possible to understand the finite-size theorem in section 5 without reading section 4, except for the conclusion of theorem 4.5. Basically, while section 4 is somewhat technical it explains why one can simply substitute the tricritical scaling form for the full generating function when near the tricritical point.

The outline of this paper is as follows. In the next section we define the IPDSAW model and follow that with a section on the general description of the scaling in the tricritical region. In section 4 we provide a precise description of the expected asymptotic properties of the generating function around the tricritical point. In section 5 we prove a fairly general theorem relating the analytic properties of the tricritical scaling function, $\hat{g}$ to the finite-size scaling function, $\hat{f}$. In section 6 we apply the theorem to the IPDSAW model and derive series and integral representations for $\hat{f}$. We end with a discussion of the relevance of these results to the polymer collapse transition in general.

2. The IPDSAW model

The IPDSAW model is a model exhibiting a collapse transition [8, 4]. It consists of a partially-directed walk with nearest-neighbour interactions. A partially-directed walk on the square lattice is a self-avoiding walk attached to the origin in which 'westerly' directed steps are forbidden. The partition function for an n step walk is

$$\mathcal{Z}_n(\omega) = \sum_{\varphi_n \in \Phi_n} \omega^{m(\varphi_n)} \tag{1}$$

where ω is the Boltzmann weight $\exp(\beta J)$, J the energy of a single nearest-neighbour interaction, β the inverse temperature, Φ_n is the set of n-step partially-directed walks and $m(\varphi_n)$ is the number of nearest-neighbour interactions in a given configuration φ_n. In the thermodynamic limit, $n \to \infty$, the model undergoes a phase transition from an extended phase for $\omega < \omega_c$ to a collapsed phase for $\omega > \omega_c$. The free energy per step, given by

$$\mathcal{F}(\omega) = \lim_{n\to\infty} -\frac{1}{n\beta} \log(\mathcal{Z}_n(\omega)) \tag{2}$$

has a singularity at $\omega = \omega_c$. If the generating function

$$G(\omega, \upsilon) = \sum_{n=1}^{\infty} \mathcal{Z}_n(\omega)\upsilon^n \tag{3}$$

considered as a function of υ has a radius of convergence $\upsilon_c(\omega)$, then it can be shown that

$$\beta\mathcal{F}(\omega) = \log(\upsilon_c(\omega)). \tag{4}$$

Mathematically it turns out to be more convenient to introduce the reduced partition function, $\mathcal{Q}_n$, and study an alternative generating function

$$\mathcal{G}(\omega, q) = G(\omega, \upsilon) = \sum_{n=1}^{\infty} \mathcal{Q}_n(\omega) q^n \qquad \mathcal{Q}_n(\omega) = \omega^{-n}\mathcal{Z}_n(\omega) \qquad q = \omega\upsilon. \tag{5}$$

This essentially allows the simple transformation to scaling variables in the IPDSAW model. (In more general models, such as ISAW, this transformation should still be analytic if more complicated—the ω^{-n} becomes some analytic function $\mu_a^{-n}(\omega)$.) A solution for $\mathcal{G}$ expressed in terms of q-Bessel functions has been found [8] (and in terms of Bessel functions for the semicontinuous case [4]), from which an extensive study of the thermodynamic properties has been made [9, 4].

3. General scaling: a brief survey

Of relevance is the radius of convergence, $q_c(\omega)$, of $\mathcal{G}$ considered as a function of q, with ω a parameter, which is shown schematically in figure 1. We shall refer to the line $q_c(\omega)$ as the 'critical line'. The most significant feature of figure 1 is the point $(\omega_c, 1)$ around which the generating function behaves mathematically in a tricritical fashion [10].

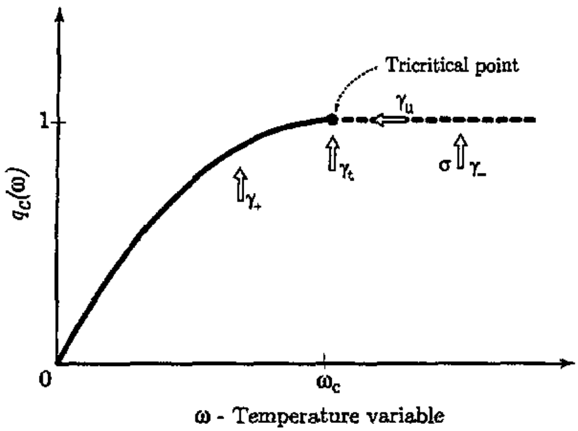

Figure 1. The radius of convergence, $q_c(\omega)$, of $\mathcal{G}(q, \omega)$ in the IPDSAW model, showing the tricritical point at $(\omega_c, 1)$. For other models the right-hand side of the curve need not run along $q = 1$. The arrows indicate the paths on which the various exponents are defined.

Before we use the asymptotic symbol '$\sim$' please note that by the expression '$g(x) \sim f(x)$ as $x \to x_c$' we will *always mean* that $\lim_{x \to x_c} g/f = 1$. This is equivalent to $g = f + o(f)$ as $x \to x_c$. Also, the expression $g(x, y) \sim f(x, y)$ as $x \to x_c$ for fixed $z = z(x, y)$ is *equivalent* to $g(x, y(x, z)) \sim f(x, y(x, z))$ considered as functions of x, and as $x \to x_c$, with the parameter z fixed. This last definition depends on the invertibility of $z(x, y)$. Geometrically, this corresponds to approaching x_c along a level curve $z(x, y) =$ constant.

In the neighbourhood of tricritical point one expects [10] the generating function to have the scaling form

$$\mathcal{G} \sim \mathcal{G}_s(\omega, q) \equiv A_t(1-q)^{-\gamma_t}\hat{g}(A_s\{1-q\}^{-\phi}\{\omega_c - \omega\}) \tag{6}$$

as $q \uparrow 1$ with the argument of $\hat{g}$ fixed at some value. It is however usual, in some vague sense, to assume that the right-hand side gives a good representation of the left-hand side for q close to 1 and ω close to ω_c: this region is sketched in figure 2. (Sometimes '=' or '$\approx$' are used or the idea of uniformity is mentioned: none of these is correct, as usually quoted—in fact non-uniformity and non-equality are *required* to give the various asymptotic regions correctly.) It is this 'scaling' sense we shall explicitly state below in section 4. In fact, we propose to introduce a new symbol '$\backsimeq$' to cover the assumptions discussed in section 4 as we believe they can be modified to cover some of the many two-variable cross-over scaling behaviours seen in various model and real systems.

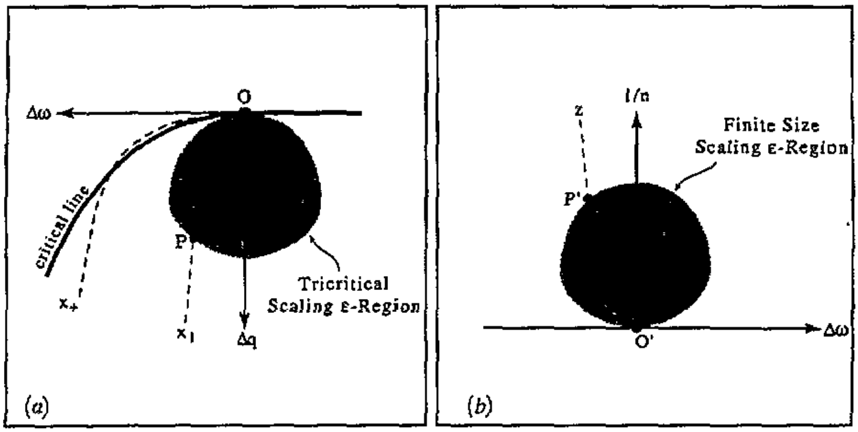

Figure 2. The region where the traditional scaling form is a good asymptotic representation of the generating function (a) and partition function (b). The shaded region in (a) is constructed by considering each level curve $A_s(1-q)^{-\phi}(\omega_c-\omega) = $ constant. If the constant is x_1 then there is a point P such that for all values of q closer to 1 the ratio of the generating function to its scaling form is closer to 1 than some preset error. The curve given by $A_s(1-q)^{-\phi}(\omega_c-\omega) = x_+$ is the place where the scaling form, $\mathcal{G}_s$, diverges—note that it does not coincide with the critical line (where the generating function diverges). The shaded region in (b) is constructed in a similar way by considering when the ratio of the partition function to the scaling form becomes close to 1 on level curves $z \equiv n^{\phi}(\omega_c-\omega) = $ constant.

The tricritical scaling function $\hat{g}$ behaves like

$$\hat{g}(x) \sim \begin{cases} G_+(x_+-x)^{-\gamma_+} & \text{as } x \uparrow x_+ > 0 \\ 1 & \text{as } x \to 0 \\ G_-(-x)^{-\gamma_t/\phi} & \text{as } x \downarrow -\infty \end{cases} \tag{7}$$

where γ_t, γ_+ and ϕ are universal exponents, and A_t, A_s, G_+ and x_+ are non-universal constants. In the IPDSAW case $\hat{g}(x)$ has a power-law essential singularity as $x \downarrow -\infty$—in other models there may be a singularity at some finite negative value x_-. Note that the *natural* scaling variable is one that is linear in the temperature difference to the tricritical point; in field theoretic discussions the mass gap or inverse correlation length often takes that role and a scaling function written in that variable will usually be manifestly non-analytic at the origin (for example see [11]). The scaling function $\hat{g}(x)$ is universal up to a multiplicative constant in the argument and an overall multiplicative constant.

In addition one expects [10] the partition function to have a finite-size scaling form

$$Q_n(\omega) \sim A_t n^{\gamma_t-1}\hat{f}(n^{\phi}\{\omega_c-\omega\}) \tag{8}$$

as $n \to \infty$ with the argument of $\hat{f}$ fixed at some value. The function $\hat{f}(z)$ is the finite-size scaling function and is universal up to a multiplicative constant in the argument and an overall factor (we have not shown these explicitly). The finite-size scaling function $\hat{f}(z)$ is expected [10] to behave like

$$\hat{f}(z) \sim \begin{cases} F_+ z^{-(\gamma_t-\gamma_+)/\phi} \mu_+^{z^{1/\phi}} & \text{as } z \to \infty \\ F_0 & \text{as } z \to 0 \\ F_- |z|^{-(\gamma_t-\gamma_-)/\phi} \mu_s^{|z|^{\sigma/\phi}} & \text{as } z \to \infty \end{cases} \tag{9}$$

where σ is the surface free energy exponent† and F_j, μ_+ and μ_s are non-universal constants.

4. Asymptotic behaviour of the scaling functions

The principal result of the next section is to prove that the finite-size scaling function is entire. To do this we will need two results: first an asymptotic form of $\mathcal{G}$ that is *uniform* throughout some fixed interval containing ω_c and second, the asymptotic form must have the same radius of convergence, $q_c(\omega)$ as the generating function itself (this is so we can apply Darboux's theorem throughout the fixed interval). Unfortunately, the traditional tricritical scaling form (6) fails *both* requirements. However, as shown in this section it can be expanded by use of two 'extending' functions. The purpose of one of the extending functions is to ensure the extended tricritical form has singularities at the desired places, whilst the second ensures the correct behaviour on the critical line. These allow a uniform asymptotic expansion to be constructed.

4.1. Tricritical scaling

In the following definition we describe mathematically our understanding of a tricritical point and the traditional scaling that is expected to hold (for an asymmetric model [10] such as the IPDSAW). Note, all the limits are taken through real values and any constants used are assumed to be non-zero. Also, all functions of ω and/or q are real valued for real values of their arguments. The partition function $Q_n(\omega)$ and $\mathcal{G}(\omega, q)$ are positive valued for all physical values of their arguments.

Definition 4.1 (tricritical scaling). Let $\mathcal{G}(\omega, q)$ generate the functions $Q_n(\omega)$,

$$\mathcal{G}(\omega, q) = \sum_{n=1}^{\infty} Q_n(\omega) q^n. \tag{10}$$

Let I_+ be the open interval (ω_0, ω_c), I_- be the open interval (ω_c, ω_1) and $I = I_+ \cup \{\omega_c\} \cup I_-$, for some ω_0, ω_1 for which $0 < \omega_0 < \omega_c < \omega_1$. Consider $\mathcal{G}(\omega, q)$ a function of q with ω a real, positive valued parameter. If the following conditions hold tricritical scaling is said to occur at $(\omega_c, 1)$:

(i) $\mathcal{G}(\omega, q)$ has a radius of convergence, $q_c(\omega) > 0$, with

$$q_c(\omega) = 1 \qquad \text{for } \omega \in I_- \cup \{\omega_c\} \tag{11}$$

$$q_c(\omega) < 1 \qquad \text{and analytic} \qquad \text{for } \omega \in I_+ \tag{12}$$

and $q_c(\omega)$ monotonically increasing for $\omega \in I_+$;

(ii) for D, $\phi \in \mathbb{R}^+$,

$$1 - q_c(\omega) \sim D(\omega_c - \omega)^{1/\phi} \qquad \text{as } \omega \uparrow \omega_c \tag{13}$$

(iii) for $\gamma_+ \in \mathbb{R}^+$ and $A_+(\omega)$ an analytic function of ω for $\omega \in I_+$, and for fixed ω,

$$\mathcal{G}(\omega, q) \sim A_+(\omega)(q_c(\omega) - q)^{-\gamma_+} \qquad \text{as } q \uparrow q_c(\omega), \omega \in I_+ \tag{14}$$

† Note, the $z \to -\infty$ behaviour may be slightly different depending on whether the model is 'symmetric' or not [10].

(iv) for A_t, $\gamma_t \in \mathbb{R}^+$,

$$\mathcal{G}(\omega, q) \sim A_t(1-q)^{-\gamma_t} \qquad \text{as } q \uparrow q_c(\omega), \omega = \omega_c \tag{15}$$

(v) for $A_-(\omega)$ an analytic function of ω for $\omega \in I_-$, and for fixed ω,

$$\mathcal{G}(\omega, q) \sim A_-(\omega) \qquad \text{as } q \uparrow 1, \omega \in I_- \tag{16}$$

(vi) for S_+, $S_- \in \mathbb{R}^+$ and $\gamma_u = \gamma_t/\phi$ the amplitudes

$$A_-(\omega) \sim S_-(\omega - \omega_c)^{-\gamma_u} \qquad \text{as } \omega \downarrow \omega_c \tag{17}$$

$$A_+(\omega) \sim S_+(\omega_c - \omega)^{\gamma_+/\phi - \gamma_u} \qquad \text{as } \omega \uparrow \omega_c \tag{18}$$

(vii) for all fixed $x \in (-\infty, x_+)$, with $x_+ > 0$,

$$\mathcal{G}\left(\omega_c - \frac{x(1-q)^\phi}{A_s}, q\right) \sim A_t \hat{g}(x)(1-q)^{-\gamma_t} \qquad \text{as } q \uparrow 1 \tag{19}$$

where $A_s = x_+ D^\phi$;

(viii) and the 'tricritical scaling function' $\hat{g}(x)$ is an analytic function for $x \in (-\infty, x_+)$ with the radius of convergence R of $\hat{g}$ equal to x_+ and a singularity on its radius of convergence at $x = x_+$, and behaves as

$$\hat{g}(x) \sim G_+(x_+ - x)^{-\gamma_+} \qquad \text{as } x \uparrow x_+ \tag{20}$$

$$\hat{g}(x) \sim G_-(-x)^{-\gamma_u} \qquad \text{as } x \downarrow -\infty \tag{21}$$

and $\hat{g}(0) = 1$, $G_+ = S_+ D^{\gamma_t - \gamma_+}(x_+\phi)^{\gamma_+}/A_t$, $G_- = S_- D^{\gamma_t} x_+^{\gamma_u}/A_t$.

We note that the asymptotic relation (19) can be rephrased into the implicit form: for fixed $x = A_s(1-q)^{-\phi}(\omega_c - \omega)$

$$\mathcal{G}(\omega, q) \sim \mathcal{G}_s(\omega, q) \equiv A_t(1-q)^{-\gamma_t}\hat{g}(A_s\{1-q\}^{-\phi}(\omega_c - \omega)) \qquad \text{as } q \uparrow 1. \tag{22}$$

We make three further points about this definition: one is that neither expression (14) nor (16) are uniform in the variable ω. They must, in fact, be manifestly non-uniform to cross-over into the form (15) at $\omega = \omega_c$. Secondly and similarly, the asymptotic expression (19) is not uniform in the parameter x. Finally, this definition concerns models like the IPDSAW which have an asymmetric tricritical transition [10]; it can be easily modified for the symmetric case. It can also be modified if say $\gamma_t < 0$ with some work.

4.2. Extended tricritical scaling function

Lemma 4.2. Let the function $\mathcal{G}(\omega, q)$ have a tricritical scaling behaviour as stated in the *definition 4.1, then there exist real valued functions $d(\omega)$ and $h(\omega)$ such that* $\mathcal{G}(\omega, q) \sim \mathcal{G}_u$, as $q \uparrow q_c(\omega)$ for all fixed $\omega \in I$ where

$$\mathcal{G}_u(\omega, q) = A_t d(\omega)(1-q)^{-\gamma_t}\hat{g}(A_s(1-q)^{-\phi}h(\omega)). \tag{23}$$

The 'extending' functions d and h are analytic for all $\omega \in I$ except possibly at $\omega = \omega_c$, continuous for all $\omega \in I$, and

$$d(\omega) \sim 1 \qquad \text{as } \omega \to \omega_c \tag{24}$$

$$h(\omega) \sim \omega_c - \omega \qquad \text{as } \omega \to \omega_c. \tag{25}$$

The proof of the theorem is straightforward as it is possible to give explicit expressions for h and d. Using these expressions and the properties of $\hat{g}$ essentially give the required result.

Proof. Let

$$h(\omega) = \begin{cases} D^{-\phi}(1 - q_c(\omega))^{\phi} & \omega \in I_+ \\ \omega_c - \omega & \omega \in \{\omega_c\} \cup I_- \end{cases} \tag{26}$$

then h is analytic for $\omega \in I_+, I_-$ since $q_c(\omega)$ is analytic for $\omega \in I_+, I_-$. Using (13) shows $h \sim \omega_c - \omega$ as $\omega_c \to \omega$ and hence h is continuous in I.

The principal function of h is to ensure that the argument of $\hat{g}$ reaches x_+ on the critical line. As q tends to $q_c(\omega)$ the argument of $\hat{g}$ increases until it reaches the singularity at x_+. Unless the argument of $\hat{g}$ is given by $A_s(1-q)^{-\phi}h(\omega)$ (this being the equation of the critical line of $\mathcal{G}$) the scaling function will reach its singularity at a different point in the ωq-plane to that of $\mathcal{G}$. Hence, by introducing h as in (23) the argument of $\hat{g}$ is controlled to ensure that the value x_+ is reached on the critical line *whatever* its shape, as is readily verified by direct substitution. Note that $h(\omega)$ is invertible in both I_+ and I_- since $q_c(\omega)$ is monotonic.

Let

$$d(\omega) = \begin{cases} (x_+\phi)^{\gamma_+} A_t^{-1} G_+^{-1} (1 - q_c(\omega))^{\gamma_t - \gamma_+} A_+(\omega) & \omega \in I_+ \\ 1 & \omega = \omega_c \\ A_s^{\gamma_u} A_t^{-1} G_-^{-1} (\omega - \omega_c)^{\gamma_u} A_-(\omega) & \omega \in I_- \end{cases} \tag{27}$$

then, in a manner similar to the h argument, using (13), (17) and (18) shows that d has the required behaviour (24).

The purpose of d is to ensure the correct amplitudes $A_+(\omega)$ and $A_-(\omega)$ are obtained. Substituting (26) and (27) into (23) and for *fixed* $\omega \in I$ and q sufficiently close to the critical line enables us to use (20) and (21), which gives the right-hand sides of (14), (15) and (16) directly. Since, by definition 4.1, $\mathcal{G}$ is also asymptotic to the right-hand sides of (14), (15) and (16) we have that $\mathcal{G} \sim \mathcal{G}_u$ as $q \uparrow q_c(\omega)$ for $\omega \in I$ and fixed. Note that $d(\omega)$ and $h(\omega)$ in I_- are not unique as we are free to choose any monotonic function $h(\omega)$ that has the property (25) after which $d(\omega)$ is then determined. □

The above theorem gives us a single function, $\mathcal{G}_u$, representing the asymptotic behaviour of $\mathcal{G}$ as the critical line is approached from below *for fixed* ω. However, this does not completely specify the asymptotic behaviour of $\mathcal{G}$. For example, what happens if the tricritical point is approached along some curve in the ωq-plane rather than from directly below? In fact, $\mathcal{G} \sim \mathcal{G}_u$ for fixed $y = A_s(1-q)^{-\phi}h(\omega)$ as $q \uparrow 1$. This is a simple consequence of the above definition and lemma.

Corollary 4.3. Let the function $\mathcal{G}(\omega, q)$ have the tricritical scaling behaviour as stated in the definition 4.1, and that the functions $h(\omega)$ and $d(\omega)$ are those defined in lemma 4.2 and its proof. Let $h^{-1}(t)$ be the inverse function of h, then

$$\mathcal{G}(\bar{\omega}(q), q) \sim A_t d(\bar{\omega}(q))(1-q)^{-\gamma_t}\hat{g}(y) \qquad \text{as } q \uparrow 1 \tag{28}$$

where

$$\bar{\omega}(q) = h^{-1}(\bar{t}(q)) \tag{29}$$

with

$$\bar{t}(q) = \frac{y(1-q)^{\phi}}{A_s} \tag{30}$$

for all fixed $y \in (-\infty, x_+)$.

We omit the proof of this result as it is a straightforward application of the properties (24) and (25) and the condition (19), and is unenlightening.

4.3. Uniform asymptotics

We wish to find a uniform asymptotic form for the generating function valid throughout the interval I as the critical line is approached along any curve. To establish this will require the introduction of one further assumption. It is based on the notion of 'asymptotic completeness'. We introduce this notion to describe the general, but somewhat vague, belief that around a tricritical point the asymptotic behaviour of $\mathcal{G}$ is *fully* represented by a fixed set of critical exponents and hence a corresponding set of functions.

In the following theorem we show that (23) is uniformly valid for all $\omega \in I$. This is based on the results of lemma 4.2 (that $\mathcal{G}_u$ gives the correct asymptotic behaviour for each fixed $\omega \in I$), on the corollary 4.3 (that it also gives the correct asymptotic behaviour for each fixed $y \in (-\infty, x_+)$ as $q \uparrow 1$ with $y = A_s(1-q)^{-\phi}h(\omega)$), and on the added assumption of 'asymptotic completeness'.

Asymptotic completeness is required for a particular technical reason. The asymptotic behaviour of lemma 4.2 and traditional scaling give rise to several domains in which the various asymptotic statements are valid. For $\mathcal{G}_u$ to hold uniformly it is necessary that these domains overlap. Without asymptotic completeness it is in general possible for thin 'wedges' to separate these various domains. It is then in principle possible for $\mathcal{G}$ to have an additional asymptotic form, and hence critical exponents, along a curve that lies within a wedge. Since it is generally believed that around a tricritical point no such 'additional' behaviour occurs we make it a condition that it does not. For any given model it may be possible to show that these wedges do not exist (i.e. the domains overlap) and so the invocation of asymptotic completeness would be unnecessary. However, as the theorems do not refer to any particular model we need to make this assumption.

The discussion is simplified if we first introduce the following definition.

Definition 4.4 (ε-asymptotic region). If $u(x) \sim p(x)$ as $x \uparrow x_0$ then the ε-asymptotic region $\Delta_p^u(\varepsilon)$ is defined as the interval below x_0 for which $|u(x)/p(x) - 1| \leqslant \varepsilon$. This interval is given as $x_0 > x > x_0 - \delta_p^u(\varepsilon)$. If the functions depend on another parameter, ω say, the region is an area in the ωx-plane and the value $\delta_p^u(\varepsilon, \omega)$ depends generically on that parameter.

The assumption of asymptotic completeness translates technically to the assumption that close to the tricritical point the ε-asymptotic region where the generating function is asymptotic to its (fixed) high/low temperature behaviour becomes small (as $\omega \to \omega_c$) no faster than one of the level curves $y = A_s(1-q)^{-\phi}h(\omega) = \text{constant}$. Basically this patches the high/low temperature asymptotic forms with the tricritical one. By construction $\mathcal{G}_u$ gives both these forms and so can be shown to be uniform.

Theorem 4.5 (Tricritical uniformity). Let $\mathcal{G}(\omega, q)$ be a generating function satisfying the *conditions stated in definition 4.1, lemma 4.2 and corollary 4.3. For $\omega \in I_+$ (resp. $\omega \in I_-$)* and fixed, denote the ε-asymptotic region for which $\mathcal{G} \sim A_+(q_c(\omega) - q)^{-\gamma_+}$(resp. $\mathcal{G} \sim A_-(\omega)$) by $\Delta_{\mathcal{G}}^+(\varepsilon)$(resp. $\Delta_{\mathcal{G}}^-(\varepsilon)$) and the ε-asymptotic region for which $\mathcal{G}_u \sim A_+(q_c(\omega) - q)^{-\gamma_+}$(resp. $\mathcal{G}_u \sim A_-(\omega)$) by $\Delta_{\mathcal{G}_u}^+(\varepsilon)$(resp. $\Delta_{\mathcal{G}_u}^-(\varepsilon)$).

(i) If for all $\omega \in I_+$ there exists $\dot{y}(\varepsilon) \in (0, x_+)$ such that

$$\delta_{\mathcal{G}}^+(\varepsilon, \omega) \geqslant \left(\frac{A_s h(\omega)}{\dot{y}(\varepsilon)}\right)^{1/\phi} - \left(\frac{A_s h(\omega)}{x_+}\right)^{1/\phi} \tag{31}$$

and

(ii) if for all $\omega \in I_-$ there exists $\ddot{y}(\varepsilon) \in (-\infty, 0)$ such that

$$\delta_{\mathcal{G}}^{-}(\varepsilon, \omega) \geqslant \left(\frac{A_s h(\omega)}{\ddot{y}(\varepsilon)}\right)^{1/\phi} \tag{32}$$

then

$$\mathcal{G} \sim A_t d(\omega)(1-q)^{-\gamma_t} \hat{g}(A_s(1-q)^{-\phi} h(\omega)) \qquad \text{as } q \uparrow q_c(\omega) \tag{33}$$

uniformly for all $\omega \in I$.

As mentioned above the proof is based on showing that the ε-asymptotic regions arising from the various asymptotic statements overlap in such a way as to give rise to a common region adjacent to the critical line. Most of these regions are illustrated schematically in figures 3 and 4 which we shall refer to throughout the following proof. We shall use $\Delta q = 1 - q$ and $\Delta\omega = \omega_c - \omega$. The function $y(\omega, q) = A_s(1-q)^{-\phi} h(\omega)$.

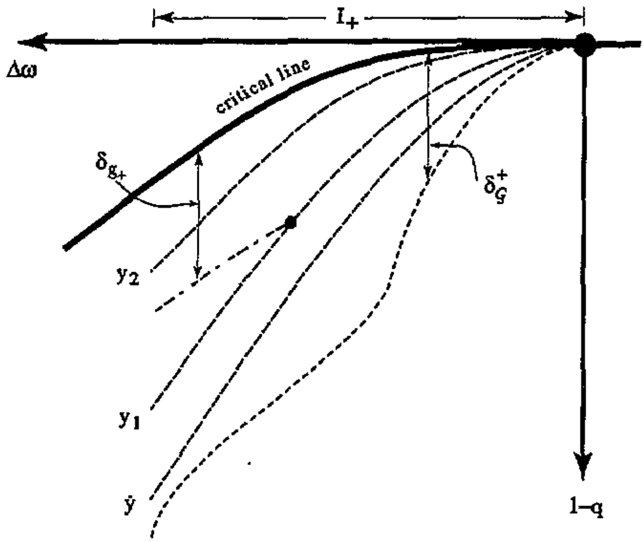

Figure 3. A schematic illustration of the various ε-asymptotic boundaries. The lines labelled by values of y are level curves of $y = A_s(1-q)^{-\phi}h(\omega)$. In this case we have chosen $\dot{y} < y_1$—it could of course be the case that $\dot{y} > y_2$. However, whatever the case, the maximum of $\dot{y}$ and y_2 is less than x_+ (that is, the critical line). The y_2 value is chosen so that it is above the y_1 line and the δ_{g_+} line.

Proof. We begin by showing that the $\Delta_{\mathcal{G}_u}^{+}(\varepsilon)$ region contains the region that lies between the level curve $y(\omega, q) = \text{constant} \equiv y_1 < x_+$, a line of constant $(q_c(\omega) - q) > 0$ and the critical line. From (20) we have an ε-asymptotic region $x_+ - y < \delta_{\hat{g}}^{\hat{g}+}(\varepsilon')$ for which $|\hat{g}(y) - \hat{g}_+(y)| < \varepsilon'|\hat{g}_+(y)| = \varepsilon'\hat{g}_+(y)$, where $\hat{g}_+(y) = G_+(x_+ - y)^{-\gamma_+}$. Thus

$$\mathcal{G}_u = A_t \Delta q^{-\gamma_t}(\hat{g}(y) - \hat{g}_+(y)) + A_t \Delta q^{-\gamma_t} \hat{g}_+(y) \tag{34}$$

$$< A_t \Delta q^{-\gamma_t} \varepsilon' |\hat{g} + +(y)| + A_t \Delta q^{-\gamma_t} |\hat{g}^+(y)| \tag{35}$$

$$= (1+\varepsilon')A_+(q_c(\omega) - q)^{-\gamma_+} + (1+\varepsilon')\mathrm{O}(\{q_c(\omega) - q\}^{-\gamma_+ + 1}) \tag{36}$$

where we have used $A_t \Delta q^{-\gamma_t} \hat{g}_+(y) = A_+(q_c(\omega) - q)^{-\gamma_+} + \mathrm{O}(\{q_c(\omega) - q\}^{-\gamma_+ + 1})$, hence

$$\mathcal{G}_u / A_+(q_c(\omega) - q)^{-\gamma_+} - 1 < \varepsilon' + (1+\varepsilon')\mathrm{O}(q_c(\omega) - q). \tag{37}$$

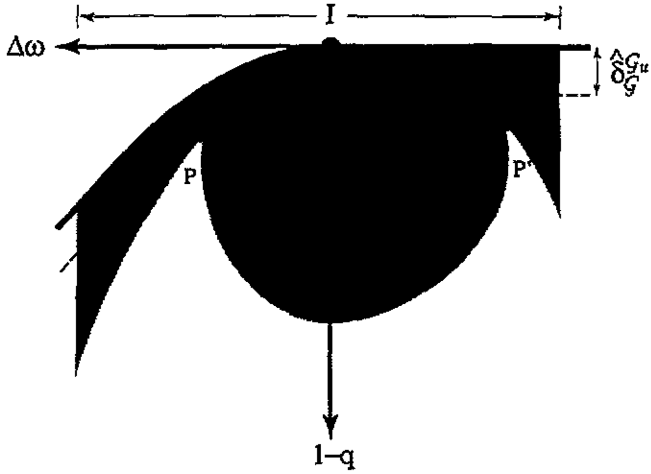

Figure 4. A schematic illustration of the region that is contained in the ε-asymptotic region $\Delta_{\mathcal{G}}^{\mathcal{G}_u}$. The wings are the regions bounded by the y_3 line (and its $\omega \geqslant \omega_c$ counterpart), while the middle is the scaling region defined by the locus of values $\bar{\delta}_{\mathcal{G}}^{\mathcal{G}_u}(y)$ which are strictly positive for $y \in [0, y_3]$. The intersection of these two regions defines the point P (and for $\omega \geqslant \omega_c$ the point P'). These points are below the critical line by construction and so allow us to choose the uniform value $\hat{\delta}_{\mathcal{G}}^{\mathcal{G}_u}$. The region defined by the $\hat{\delta}_{\mathcal{G}}^{\mathcal{G}_u}$ curve and the critical line is then certainly contained in $\Delta_{\mathcal{G}}^{\mathcal{G}_u}$.

Thus, if we choose $\varepsilon' = (\varepsilon - \mathrm{O}(q_c(\omega) - q))/(1 + \mathrm{O}(q_c(\omega) - q))$ and q sufficiently close to $q_c(\omega)$, say $q_c(\omega) - q = \delta_{g_+}$, so that $\varepsilon' > 0$, then $\mathcal{G}_u \sim A_+(q_c(\omega) - q)^{-\gamma_+}$ as $q \uparrow q_c(\omega)$ with an ε-asymptotic region given by the intersection of $q \in (q_c(\omega) - \delta_{g_+}, q_c(\omega))$ and $x_+ - y(\omega, q) < \delta_{\hat{g}}^{\hat{g}_+}(\varepsilon')$, for each $\omega \in I_+$. Given that I_+ is a fixed interval in ω it is easy to see (figure 3) that one can now choose some $y_2 \in (y_1, x_+)$ such that the region lying between $y(\omega, q) = y_2$ and the critical curve is certainly contained in $\Delta_{\mathcal{G}_u}^{+}(\varepsilon)$.

From (31) we have that the ε-asymptotic region for $\mathcal{G} \sim A_+(q_c(\omega) - q)^{-\gamma_+}$ is at least the region defined by the critical curve and the curve $y(\omega, q) = \dot{y}$, $\omega \in I_+$. One can now simply choose (see figure 3) the maximum of y_2 and $\dot{y}$, say y_3, so that the region defined by the area between the critical curve and the curve $y(\omega, q) = y_3$ is certainly contained in the $\Delta_{\mathcal{G}}^{\mathcal{G}_u}(3\varepsilon)$ ε-asymptotic region.

Now, the ε-asymptotic region $\Delta_{\mathcal{G}}^{\mathcal{G}_u}(\varepsilon)$ is also defined by considering the asymptotics at fixed y. From corollary 4.3, for each fixed $y \in [0, x_+)$ there exists a $\bar{\delta}_{\mathcal{G}}^{\mathcal{G}_u}(\varepsilon, y)$ such that $|\mathcal{G}/\mathcal{G}_u - 1| < \varepsilon$ for $|1 - q| < \bar{\delta}_{\mathcal{G}}^{\mathcal{G}_u}(\varepsilon, y)$. The value $\bar{\delta}_{\mathcal{G}}^{\mathcal{G}_u}(\varepsilon, y)$ is non-zero for all $y \in (0, x_+)$ by definition. The worst that can happen is that it tends to zero as $y \uparrow x_+$ (see figure 4). It is however strictly positive for $y \in [0, y_3]$. By considering figure 4 it is clear that the two different regions described above, both of which are contained within the ε-asymptotic region $\Delta_{\mathcal{G}}^{\mathcal{G}_u}(\varepsilon)$, share a common neighbourhood. It is then clear that one can choose some $\hat{\delta}_{\mathcal{G}}^{\mathcal{G}_u}(\varepsilon)$ *independent* of ω such that the region defined by $|q_c(\omega) - q| < \hat{\delta}_{\mathcal{G}}^{\mathcal{G}_u}(\varepsilon)$ is wholly contained in the $\Delta_{\mathcal{G}}^{\mathcal{G}_u}(\varepsilon)$ region for I_+. (If there is another line $y_4 = y(\omega, q)$, with $y_4 \in [0, y_3]$, whose ε region begins closer to the critical line one simply chooses that point to give $\hat{\delta}_{\mathcal{G}}^{\mathcal{G}_u}(\varepsilon)$.) Hence the asymptotic relation $\mathcal{G}(\omega, q) \sim \mathcal{G}_u(\omega, q)$ as $q \uparrow q_c(\omega)$ is uniform in $\omega \in I_+$.

These arguments apply *mutatis mutandis* for $\omega \in I_-$. □

We can now summarize all the scaling behaviour described above in definition 4.1, lemma 4.2, corollary 4.3 and theorem 4.5 by introducing the symbol '$\circledast$'. We then read

$$\mathcal{G}(\omega, q) \circledast A_t(1-q)^{-\gamma_t}\hat{g}(A_s\{1-q\}^{-\phi}\{\omega_c - \omega\}) \tag{38}$$

as '$\mathcal{G}(\omega, q)$ scales as $A_t(1-q)^{-\gamma_t}\hat{g}(A_s\{1-q\}^{-\phi}\{\omega_c - \omega\})$' which means that extending functions can be found to make a uniform asymptotic expansion; that is, asymptotic completeness is assumed.

5. Analyticity of the finite-size scaling function

In the theorem proven below we show that the finite-size scaling function of the partition function exists and moreover is entire given that the tricritical scaling function of the generating function exists and has a finite radius of convergence. To complete the list of sufficient conditions that we have compiled we add the singularity structure of $\mathcal{G}$ and $\mathcal{G}_u$ in the complex q plane. This condition is one of the usual Darboux conditions which would normally hold so that it was possible to find the behaviour of the coefficients of a power series from an asymptotic representation. It is possible that this condition could be weakened without changing the consequences.

Theorem 5.1. Let $\mathcal{G}(\omega, q)$ be a generating function satisfying the conditions stated in definition 4.1 and theorem 4.5. Let $\mathcal{G}_u$ be the uniform asymptotic behaviour of $\mathcal{G}$ in the interval $\omega \in I$. Let $\mathcal{G}^{(m)}$ and $\mathcal{G}_u^{(m)}$ be the mth derivatives. If on the circle $|q| = q_c(\omega)$ in the complex q-plane, $\mathcal{G}^{(m)} - \mathcal{G}_u^{(m)}$ has a finite number of singularities and at each singularity q_j say,

$$\mathcal{G}^{(m)} - \mathcal{G}_u^{(m)} = \mathrm{O}(\{q_j - q\}^{\sigma_j - 1}) \qquad q \to q_j \tag{39}$$

for some $m \geqslant 1$, where σ_j is some assignable positive constant, then

$$\mathcal{Q}_n(\omega) = \frac{1}{2\pi \mathrm{i}} \oint_C \mathcal{G}_u(\omega, q) \frac{\mathrm{d}q}{q^{n+1}} + o(q_c(\omega)^{-n} n^{-m}) \qquad \text{as } n \to \infty \tag{40}$$

where C encircles the origin and contains no singularities of $\mathcal{G}_u$.

Furthermore, the finite-size scaling function

$$\hat{f}(z) = \lim_{n\to\infty} \frac{\mathcal{Q}_n(\omega_c - z n^{-\phi})}{A_t n^{\gamma_t - 1}} \tag{41}$$

for any $z \in \mathbb{R}$, exists, and is given by

$$\hat{f}(z) = \sum_{m=0}^{\infty} \frac{\hat{g}^{(m)}(0)}{m!} \frac{A_s^m}{\Gamma(m\phi + \gamma_t)} z^m \tag{42}$$

where $\hat{g}^{(m)}(0)$ is the mth derivative of tricritical scaling function $\hat{g}(x)$ at $x = 0$. Finally, the function $\hat{f}$ is entire.

An outline of the proof is as follows. There are essentially three parts of the proof. The first part invokes Darboux's theorem to prove (40). After a rearrangement of the contour integral and the use of the definition of the finite-size scaling function, the second part relies on interchanging a sum and an integral, whilst the third part relies on interchanging a limit and a sum. For each of the latter two parts we need to prove a uniformity condition. These allow us to then use two theorems which are given in the appendix (see (A.2) and (A.3)).

Proof. To prove (40) we need to satisfy the conditions of Darboux's theorem (given in the appendix (A.1) for convenience). First, $\mathcal{G}_u$ and $\mathcal{G}$ are analytic in the annulus $0 < |q| < q_c(\omega)$. $\mathcal{G}$ is analytic because $q_c(\omega)$ is the radius of convergence of $\mathcal{G}$ and $\mathcal{G}_u$ is analytic because the uniforming function $h(\omega)$, as given in theorem 4.2, was constructed to ensure the radius of convergence of $\mathcal{G}_u$ coincided with $q_c(\omega)$ for $\omega \in I$. The remaining condition is satisfied by assumption (i.e. equation (39)).

Let $1 - q = t/n$, then equation (40) gives

$$Q_{n-1}(\omega) = \frac{A_t d(\omega)}{2\pi \mathrm{i}}(n-1)^{\gamma_t - 1} \oint_{C_n} t^{-\gamma_t} \left(1 - \frac{t}{n}\right)^{-n} \hat{g}\left(A_s \left(\frac{n}{t}\right)^{\phi} h(\omega)\right) \mathrm{d}t + o(q_c(\omega)^{-n} n^{-m}) \tag{43}$$

as $n \to \infty$, where C_n is a contour encircling the integer n, and hence

$$Q_{n-1}(\omega) = A_t d(\omega)(n-1)^{\gamma_t - 1}\mathcal{I}_n(z) + o(q_c(\omega)^{-n} n^{-m}) \tag{44}$$

where

$$\mathcal{I}_n(z) = \frac{1}{2\pi \mathrm{i}} \oint_{C_n} t^{-\gamma_t} \left(1 - \frac{t}{n}\right)^{-n} \hat{g}\left(A_s \frac{z}{t^{\phi}}\right) \mathrm{d}t \qquad z = n^{\phi} h(\omega). \tag{45}$$

By noting $q_c(\omega_c - zn^{-\phi})^{-n} \to$ constant as $n \to \infty$ it is not too difficult to see that the finite-size scaling function, $\hat{f}(z)$, is given by

$$\hat{f}(z) = \lim_{\substack{n \to \infty \\ z \text{ fixed}}} \mathcal{I}_n(z). \tag{46}$$

Now, we have assumed that the tricritical scaling function, $\hat{g}(x)$, has a radius of convergence $R = x_+$, where $0 < R < \infty$, and so the Taylor series

$$\sum_{m=0}^{\infty} \hat{g}_m^0 x^m \tag{47}$$

converges for $|x| \leqslant R$. Here $\hat{g}_m^0 = \hat{g}^{(m)}(0)/m!$. Substituting into (45) gives

$$\mathcal{I}_n(z) = \frac{1}{2\pi \mathrm{i}} \oint_{C_n} t^{-\gamma_t} \left(1 - \frac{t}{n}\right)^{-n} \sum_{m=0}^{\infty} \hat{g}_m^0 \left(A_s \frac{z}{t^{\phi}}\right)^m \mathrm{d}t. \tag{48}$$

The sum and the integral can only be interchanged if the sum is uniformly convergent with respect to t, and the function $t^{-m\phi}$ is continuous on the path of integration. The latter is true as the contour does not pass through or around the origin. We now prove the former condition using the Weierstrass M-test.

We need to show that the sum is uniformly convergent in a compact domain containing the contour: that is, parametrizing the contour by $t = n + \varepsilon \exp(\mathrm{i}\theta)$, we need to show uniform convergence in the domain $|t - n| \leqslant \varepsilon$. Now, choosing a number r such that $0 < \varepsilon < r < n$ we have

$$\left|\hat{g}_m^0 \left(A_s \frac{z}{t^{\phi}}\right)^m\right| \leqslant \left|\hat{g}_m^0 \left(A_s \frac{z}{(n-r)^{\phi}}\right)^m\right| \tag{49}$$

in this domain. Hence, the sum in (48) is bounded by the sum $\sum_{m=0}^{\infty} |\hat{g}_m^0 \{A_s z(n-r)^{-\phi}\}^m|$ which, by assumption, converges for $|z| < |R(n-r)^{\phi}/A_s|$. As this bounding series is independent of t, by the Weierstrass M-test the sum in (48) is uniformly convergent so long as $|z| < |R(n-r)^{\phi}/A_s|$. Thus the sum and the integral may be interchanged giving

$$\mathcal{I}_n(z) = \frac{1}{2\pi \mathrm{i}} \sum_{m=0}^{\infty} \hat{g}_m^0 (A_s z)^m \oint_{C_n} t^{-\gamma_t - m\phi} \left(1 - \frac{t}{n}\right)^{-n} \mathrm{d}t. \tag{50}$$

The integral can now be evaluated explicitly by a residue calculation, giving

$$\mathcal{I}_n(z) = \sum_{m=0}^{\infty} \hat{g}_m^0 (A_s z)^m n^{1-m\phi-\gamma_t} \frac{\Gamma(n-1+m\phi+\gamma_t)}{\Gamma(m\phi+\gamma_t)\Gamma(n)}. \tag{51}$$

We need to evaluate

$$\lim_{\substack{n\to\infty \\ z \text{ fixed}}} \mathcal{I}_n(z) \tag{52}$$

which can be done if we can interchange the limit and the sum. To do this interchange we first note that using the 'root test' one can show that for any $|z| < |R(n-r)^\phi/A_s|$ the right-hand side of (51) is absolutely convergent. One can also show using the Weierstrass M-test that the series is convergent uniform in $n > N$ for $|z| < |R(N-r)^\phi/A_s|$. It can also be shown that

$$n^{1-m\phi-\gamma_t} \frac{\Gamma(n-1+m\phi+\gamma_t)}{\Gamma(n)} \to 1 \qquad \text{as } n \to \infty \tag{53}$$

and that the sum

$$\mathcal{I}_\infty(z) = \sum_{m=0}^{\infty} \hat{g}_m^0 \frac{(A_s z)^m}{\Gamma(m\phi+\gamma_t)} \tag{54}$$

converges for any value z (see below). Hence for any value of $|z| < |R(N-r)^\phi/A_s|$ we can now use the theorem (A.3) of the appendix: a sequence of uniformly convergent series will converge to a sum of limiting terms given that the limit series is also convergent. Note that as the bound on z can be made arbitrarily large we have shown that we can accomplish the interchange for any z.

Thus we can indeed interchange the limit and the sum in (52), and hence

$$\hat{f}(z) = \lim_{n\to\infty} \mathcal{I}_n(z) = \sum_{m=0}^{\infty} \hat{g}_m^0 (A_s z)^m \lim_{n\to\infty} n^{1-m\phi-\gamma_t} \frac{\Gamma(n-1+m\phi+\gamma_t)}{\Gamma(m\phi+\gamma_t)\Gamma(n)}. \tag{55}$$

Evaluating the limit gives

$$\hat{f}(z) = \mathcal{I}_\infty(z) = \sum_{m=0}^{\infty} \hat{g}_m^0 \frac{A_s^m}{\Gamma(m\phi+\gamma_t)} z^m. \tag{56}$$

Finally, considering $\sum_{m=0}^{\infty} |\hat{g}_m^0 (A_s z)^m / \Gamma(m\phi+\gamma_t)| =: \sum_{m=0}^{\infty} a_m z^m$ as a power series in z then the radius of convergence, R_a is given by $1/R_a = \lim\sup_{m\to\infty} a_m^{1/m}$. We now show that this limit is zero and hence R_a is infinite. Since any convergent power series also converges absolutely and $\hat{g}(x) = \sum_{m=0}^{\infty} \hat{g}_m^0 x^m$ is assumed to converge with radius of convergence R, we have that

$$1/R = \lim_{m\to\infty} \sup |\hat{g}_m^0|^{1/m}. \tag{57}$$

Thus

$$\begin{aligned} 1/R_a &= \lim_{m\to\infty} \sup |\hat{g}_m^0 A_s^m / \Gamma(m\phi+\gamma_t)|^{1/m} \\ &= \lim_{m\to\infty} \sup |\hat{g}_m^0|^{1/m} \lim_{m\to\infty} \sup |A_s^m / \Gamma(m\phi+\gamma_t)|^{1/m} \\ &= 0 \end{aligned} \tag{58}$$

and hence $R_a = \infty$. Since the right-hand side of (56) has an infinite radius of convergence it is an entire function. □

6. Finite-size scaling and the IPDSAW

For the (semi-continuous) IPDSAW model it has been shown [4] that the generating function is indeed of the form (6), with

$$\gamma_t = \frac{1}{3} \qquad \phi = \frac{2}{3} \qquad A_t = \left(\frac{\beta_c J}{2}\right)^{1/3} \qquad A_s = \frac{1}{\omega_c (4\beta_c J)^{1/3}} \tag{59}$$

remembering that $\omega = e^{\beta J}$, and the tricritical scaling function $\hat{g}_I$ is given by

$$\hat{g}_I(x) = -\frac{\mathrm{Ai}(-x)}{\mathrm{Ai}'(-x)} \tag{60}$$

where $\mathrm{Ai}(\cdot)$ is the Airy function. The only differences in the fully discrete case are the non-universal constants A_t and A_s. Given that the semicontinuous model has been studied more extensively we use it for convenience (the corresponding theorem is a straightforward modification and gives the same result).

As the Airy function and its derivative are entire functions and non-zero in a neighbourhood of the origin, it implies that $\hat{g}_I$ has a finite radius $R = a_0$ where $a_0 = 1.018\,79\ldots$ is the closest zero to the origin of $\mathrm{Ai}'(-x)$. Thus by the above theorem, $\hat{f}_I$ must be an entire function, given by

$$\hat{f}_I(z) = \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{\mathrm{d}^m}{\mathrm{d}x^m} \hat{g}_I(x) \mid_{x=0} \right) \frac{z^m}{\omega_c^m (4\beta_c J)^{m/3} \Gamma((2m+1)/3)}. \tag{61}$$

We shall now deduce the asymptotic behaviour of $\hat{f}_I$ in the two cases $z \to \pm\infty$. First however, we see that for $z = 0$ it is simple to find $\hat{f}_I(0) = F_0 = 3^{-1/3}/\Gamma(2/3)$.

For $z \ll 0$ it is not useful to work directly with (61) but we can deduce the $z \to -\infty$ behaviour from the results already obtained in [12] where it has been shown that, for fixed $\omega > \omega_c$,

$$Z_n(\omega) \sim H_1 \omega^n \exp(-H_2 n^{1/2}) n^{-3/4} \qquad n \to \infty, \omega > \omega_c \tag{62}$$

where $H_1 = (2\beta J h/\pi^2 p^4)^{1/4}$, $H_2 = \sqrt{8\beta J h}$, $h = \log((1+p)/\sqrt{\beta_c/\beta}) - p$ with $p = \sqrt{1 - \beta_c/\beta}$. For small $\Delta\omega$ we can rearrange the right-hand side of (62) into the form

$$K_1 n^{-2/3} \omega^n \left(\frac{\Delta\omega n^{2/3}}{\omega_c (4\beta_c J)^{1/3}} \right)^{-1/8} \exp\left[K_2 \left(\frac{\Delta\omega n^{2/3}}{\omega_c (4\beta_c J)^{1/3}} \right)^{3/4} \right] \tag{63}$$

where

$$K_1 = \frac{2^{2/3}}{(3\pi^2)^{1/4}} (\beta_c J)^{1/12} \qquad K_2 = -\left(\frac{16}{3}\right)^{1/2}. \tag{64}$$

Comparing (63) with (8) implies

$$\hat{f}_I(z) \sim F_- |z|^{-1/8} \mu_s^{|z|^{3/4}} \tag{65}$$

with $\mu_s = \exp[K_2 (4\beta_c J \omega_c^3)^{-1/4}]$ and $F_- = (4\beta_c J \omega_c^3)^{1/24} (K_1/A_t)$.

For $z \gg 0$ we use the integral representation (valid for all z)

$$\hat{f}_I(z) = \frac{1}{2\pi i} \int_C e^t t^{-1/3} \hat{g}_I \left(\frac{A_s z}{t^{2/3}} \right) \mathrm{d}t \tag{66}$$

where C is a Hankel contour from $-\infty$ around all the singularities and back in the upper half plane to $-\infty$. This is obtained by using the integral representation of the inverse of the gamma function. The large positive z behaviour can now be related to that of the Mittag–Leffler function

$$E_{2/3}(u) = \frac{1}{2\pi \mathrm{i}} \int_C \frac{\mathrm{e}^t t^{-1/3}}{t^{2/3} - u} \, \mathrm{d}t \sim \tfrac{3}{2} \exp(u^{3/2}) \tag{67}$$

(see [13]) by considering the poles of $\hat{g}_l(x)$. The constants $F_+ = 3A_\mathrm{s}/2a_0^3$ and $\mu_+ = \exp[(A_\mathrm{s}/a_0)^{3/2}]$. This agrees with the work of Louchard [14, 15] where a similar problem arises in the problem of Brownian excursion area.

Putting these results together gives

$$\hat{f} \sim \begin{cases} F_+ z \mu_+^{z^{3/2}} & \text{as } z \to \infty \\ 3^{-1/3}/\Gamma(2/3) & \text{as } z \to 0 \\ F_- |z|^{-1/8} \mu_\mathrm{s}^{|z|^{3/4}} & \text{as } z \to -\infty \end{cases} \tag{68}$$

which agrees with (9).

7. Conclusions

We have shown that in the case of the IPDSAW model in two dimensions that the finite-size scaling function is entire. Furthermore we have explicit representations of that function in terms of a Taylor series and a contour integral. To apply the above theorem to other models we need to remember that the scaled partition function $Q_n(\omega)$ was obtained from the original partition function $Z_n(\omega)$ by changing variables in the associated generating functions so that one scaling axis lies along the line $q = 1$ (with q conjugate to n for Q_n). Depending on the value of the cross-over exponent ϕ this should be a polynomial transformation and it gives rise to the function $\mu_a(\omega)$ mentioned in section 2. This allows us to move onto the question of when the conditions of the theorem should hold. We would expect that so long as the thermodynamic transition is dominated by fluctuations, so that hyperscaling holes and there is 'only one (thermal) length scale' (see for example [16–18] and references therein), such as is understood to occur below the upper critical dimension ($d < 3$ in this case), then the tricritical scaling form for the generating function should exist, and hence our theorem holds. We note that the upper critical dimension $d = 3$ is a special case and strictly speaking scaling breaks down [19] (due to the presence of logarithmic corrections) at tricritical points—this also seems to be the case in De Gennes three-dimensional θ-point description [20].

Another way to look at the scaling hypothesis is through the Yang–Lee mechanism for phase transition. The approach of the zeros of the partition function to the real temperature axis (also known as Fisher zeros) should be described by scaling [21, 22] in this case, through the cross-over exponent. That means the position of the closest zeros to the critical point should scale as

$$\omega_{\mathrm{zero}} - \omega_\mathrm{c} \sim C n^{-\phi} \tag{69}$$

(cf equation (2.17) in [21] and equation (2.11) of [22] noting that $1/\phi = 2 - \alpha$). This implies that in the limit of large n the zeros of the associated scaling function stay a fixed distance from the real axis. This also implies that the singularities of the free-energy scaling function stay a fixed distance away from the real axis. Hence, one way to test scaling (and hence the analyticity of the scaling function) is to consider the approach of the complex-temperature-plane partition-function zeros at the critical point. We also mention that Glasser *et al* [22] have calculated some finite-size scaling functions at the multicritical points of infinite-range spin models (here the size is the volume, rather than the length of walk) and furthermore that these are entire. Implicitly we assume that the free energy exists for all temperatures around the tricritical point. We note that this may not be that case for the Domb–Joyce model [23] with attractive interactions†.

The results of our theorem can be compared to the prediction [6] that, for the (two-parameter) Edwards model in dimension $d > 2$, and also for the scaling theory for the collapse transition (argued by Sokal [6] to be given by the (two-parameter) Edwards model [6], at least for repulsive interactions, in $3 < d < 4$), there are singularities in the finite-size scaling functions of various quantities on the real axis. Given that the (two-parameter) Edwards model is not a model of collapse for $d < 3$, where we argue our theorems should hold, there is no immediate contradiction in these dimensions.

We point out that our theorem holds not only for the partition functions of the IPDSAW model but also for the horizontal end-to-end displacement (see equation (3.46) of [4]) in that model. We conclude by finding that the analyticity properties of scaling functions for polymer collapse in general dimensions is an interesting open question.

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Appendix A. Auxiliary theorems

We state Darboux's theorem as given in Olver [24].

Theorem A.1 (Darboux 1878). Let $\omega(t)$ be a given analytic function and

$$\omega(t) = \sum_{n=-\infty}^{\infty} a_n t^n \tag{70}$$

its Laurent expansion in an annulus $0 < |t| < r$. Let $c(t)$ be a function with the following properties:

(i) $c(t)$ is analytic in $0 < |t| < r$.

(ii) On the circle $|t| = r$, the difference of the mth derivatives ($m \geqslant 0$), $\omega^{(m)}(t) - c^{(m)}(t)$ has a finite number of singularities and at each singularity t_j, say,

$$\omega^{(m)}(t) - c^{(m)}(t) = \mathrm{O}(\{t - t_j\}^{\sigma_j - 1}) \qquad t \to t_j \tag{71}$$

where σ_j is an assignable positive constant.

(iii) The coefficients b_n in the Laurent expansion

$$c(t) = \sum_{n=-\infty}^{\infty} b_n t^n \qquad 0 < |t| < r \tag{72}$$

have known asymptotic behaviour, then

$$a_n = b_n + o(r^{-n} n^{-m}) \qquad n \to \infty. \tag{73}$$

† The Domb–Joyce model with attractive interactions is characterized by a 'dot' or 'trapped' phase where the average extent of the associated walks are bounded in space, so that $\nu = 0$ (this phase is somewhat misleadingly called collapsed in the literature: collapse is used for ISAW and real polymers at low temperatures where $\nu = 1/d$). Hence, the Domb–Joyce model with attractive interactions describes the transition from a random walk to a trapped phase—this clearly has *nothing* to do with polymer physics in any finite dimension.

Theorem A.2 (Interchanging an integral and a sum). Let

$$\mathcal{S}(z) = \sum_{m=0}^{\infty} u_m(z) \tag{74}$$

be an infinite series. If the terms of (74) are continuous functions of z in some compact domain, $\mathcal{D}$ of the complex plane, and the series converges uniformly in $\mathcal{D}$ then

$$\int_{\mathcal{C}} \mathcal{S}(z)\,\mathrm{d}z = \sum_{m=0}^{\infty} \int_{\mathcal{C}} u_m(z)\,\mathrm{d}z \qquad \mathcal{C} \in \mathcal{D}. \tag{75}$$

The proof may be found in any standard textbook (e.g. [25]).

Theorem A.3 (Interchanging a limit and a sum). Let $\mathcal{S}_n = \sum_{k=0}^{\infty} a_k^{(n)}$ be a convergent series, convergent uniformly in n, and n an integer, such that $\lim_{n\to\infty} a_k^{(n)} = a_k^{\infty}$ exists, and $\mathcal{S}_\infty = \sum_{k=0}^{\infty} \lim_{n\to\infty} a_k^{(n)} = \sum_{k=0}^{\infty} a_k^{\infty}$ is a convergent series. Then it can be shown

$$\lim_{n\to\infty} \mathcal{S}_n = \mathcal{S}_\infty. \tag{76}$$

The proof is a straightforward extension of the theorem on continuity of uniformly convergent series in some parameter which can also be found in standard textbooks.

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